QCD-Inspired Spectra from Blue's Functions

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We use the law of addition in random matrix theory to analyze the spectral distributions of a variety of chiral random matrix models as inspired from QCD whether through symmetries or models. In terms of the Blue's functions recently discussed by Zee, we show that most of the spectral distributions in the macroscopic limit and the quenched approximation, follow algebraically from the discontinuity of a pertinent solution to a cubic (Cardano) or a quartic (Ferrari) equation. We use the end-point equation of the energy spectra in chiral random matrix models to argue for novel phase structures, in which the Dirac density of states plays the role of an order parameter.

1. Lattice simulations have established that QCD undergoes a phase transition at finite temperature [1]. The precise nature and character of the transition are still debated. The simulations, however, show that the QCD phase diagram with three flavors is rather involved, with a strong dependence on the value of the current masses.

Important insights into the role played by chiral symmetry may be found in the quark density of states at zero virtuality. Banks and Casher [2] have shown that at this point the density of quark states is directly proportional to the quark condensate in the chiral limit. The spontaneous breaking of chiral symmetry is followed by a structural change in the quark eigenvalues near zero virtuality. It was also observed that some generic spectral oscillations take place in this region [3,4], reminiscent of Airy oscillations [5].

An important question in QCD concerns the structure and character of the Dirac spectrum during a phase transition, as induced by changing the number of flavors, the character of the color representation, the quark masses, the temperature or the baryon density, to cite a few. Since chiral symmetry breaking is encoded in the eigenvalue distribution at zero virtuality, it is clear that a chiral phase transition would affect quantitatively this distribution. Since massive QCD does not have, strictly speaking, a good order parameter, it is natural to study the behavior of the spectral density near zero virtuality as a potential alternative. The possible change in the quark wavefunctions at zero virtuality is reminiscent of a phase transition from a delocalized and hence coherent phase, to a localized and hence incoherent phase. The coherent phase will be identified with the Goldstone phase in our case.

An efficient way to get the Dirac spectrum has been established through the use of chiral random matrix models [3,6]. Chiral random matrix models are 0-dimensional field theories, inspired by the flavor and spin structure of QCD motivated Dirac operators. They share much in common with "two-level" Nambu-Jona-Lasinio models [7]. By restricting the discussion to only constant modes, the role of symmetry and the lore of randomness become dominant and transparent. Results from random matrix models both in the microscopic [8] and macroscopic [4] limit have compared favorably with some lattice simulations of the QCD spectra [9,10].

The purpose of this letter is to use newly developed methods by Zee [11] regarding the addition laws of random matrix theory, to discuss the generic character of the Dirac spectrum in the macroscopic limit and quenched approximation. For a large class of chiral random matrix models as inspired by QCD either in the continuum or on the lattice, and instanton models, we show that the resulting spectra follow simply from the general lore of random plus deterministic. In section 2, we introduce the concept of Blue's functions and review Zee's argument for the deterministic plus random (Gaussian) case. In section 3, we show how the Blue's functions can be used to condition the end-points of the spectral distributions, resulting into quadratic equations for positive end-points (Cardano) or cubic equations for positive endpoints (Ferrari). In section 4 we apply these results to massive Wilson fermions in the one-flavor approximation, as well as massless QCD on the cylinder. The resulting spectra and singularities belong to the Cardano class. We show that the spectral distribution of QCD with 2+1 flavors in the Wilson formulation, follows from a linear superposition of the solutions to Cardano class. Instanton models with off-diagonal order fall into the Ferrari class. Our conclusions and recommendations are summarized in section 5.

2. The spectrum of a single random matrix was determined long ago by Wigner and others [12]. Generically, the averaged level distribution of an $N\times N$ random matrix H (Hamiltonian) with a probability distribution

$$P(H)=\frac{1}{Z}\exp[-N\mathrm{Tr}V(H)] \tag{1}$$

follows from the discontinuity of the resolvent

$$\nu(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \mathrm{Im} G(z = \lambda + i\epsilon) \tag{2}$$

with

$$G(z) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - H} \right\rangle \tag{3}$$

The average $< \ldots >$ is short for averaging with the weight (1). For polynomial potentials the explicit form of (3) is known [13]. In the simplest case of a quadratic polynomial, the random ensemble is known as Gaussian with the Green's function

$$G(z) = \frac{1}{2}(z - i\sqrt{4 - z^2}) \tag{4}$$

The discontinuity of (4) is just Wigner's semi-circle for Gaussian ensembles,

$$\nu(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2} \ . \tag{5}$$

The problem of adding random matrices can be reduced by using Blue's functions, as recently discussed by Zee [11]. The Blue's function $B(z)$ is just the functional inverse of the Green's function $G(z)$. Operationally

$$B(G(z)) = G(B(z)) = z \tag{6}$$

The Blue's function for a constant matrix c (random matrix of infinite weight) follows from the corresponding Green's function

$$G(z) = \frac{1}{z - c} \tag{7}$$

through the substitution $z \to B(z)$ in (6)

$$B(z) = c + \frac{1}{z} \tag{8}$$

Similarly, the Blue's function for the Gaussian ensemble with the Green's function (4) follows from (6) in the form

$$B(z) = z + \frac{1}{z} \tag{9}$$

If $B_1(z)$ and $B_2(z)$ are the Blue's functions of two random matrices, then the Blue's function $B_{1+2}(z)$ for the sum follows from the addition law [11]

$$B_{1+2}(z) = B_1(z) + B_2(z) - \frac{1}{z} \tag{10}$$

Hence, the problem of finding the spectral distribution of the sum of two random matrices boils down to the following algorithm : First, construct the appropriate Green's functions G_1 and G_2. Second, find their functional inverses, i.e. Blue's functions B_1 and B_2. Third, construct B_{1+2} using (10). Fourth, functionally invert B_{1+2} using the definition of the Blue's function, leading to G_{1+2}. The spectral density of the sum is the discontinuity of G_{1+2} along the real axis.

To illustrate these points, consider with Zee [11] the case where H is the sum of a deterministic matrix H_D with eigenvalues ϵ_i, $(i = 1, \ldots, N)$ and a random Gaussian matrix H_R. The Green's function for the deterministic matrix H_D reads

$$G_D(z) = \frac{1}{N} \sum_i \frac{1}{z - \epsilon_i} \tag{11}$$

Through the replacement $z \to B(z)$ the Blue's function B_D is

$$z = \frac{1}{N} \sum_i \frac{1}{B_D(z) - \epsilon_i} \tag{12}$$

The addition rule gives

$$B(z) = B_D(z) + B_R(z) - \frac{1}{z} = B_D(z) + z \tag{13}$$

where the second equality comes after we have used the known form of the Blue's function for the random Gaussian ensemble (9). Substituting $z \to G(z)$ in (12) and (13) we get a pair of equations

$$G(z) = \frac{1}{N} \sum_i \frac{1}{B_D(G(z)) - \epsilon_i}$$

$$z = B_D(G(z)) + G(z) \tag{14}$$

from which we eliminate $B_D(G)$[1] to get

$$G(z) = \frac{1}{N} \sum_i^N \frac{1}{z - G(z) - \epsilon_i} \tag{15}$$

This result was first established by Pastur [14]. Zee's argument adds to its transparency. We also note that (15) is nothing but the resummed rainbow diagrams [15]. It has the suggestive form of a gap equation

$$G(z) = \frac{1}{N} \mathrm{Tr} \left(\frac{1}{z - G(z) - H_D} \right) \tag{16}$$

familiar from mean-field (large N) approaches. In 0-dimensional field theories and for a Gaussian measure, the self-energy reduces to the two-point function. Pastur's equation is a polynomial in G of degree $p + 1$. Algebraic solutions are only available for $p \leq 3$. Below we will show that a variety of QCD inspired random matrix models fulfills this restriction, with a chiral condensate in the massless case given by

[1] Note that the explicit form of $B_D(z)$ is actually not needed.

$$< \bar{q}q >= \text{Im } G(z = i\epsilon) \ . \tag{17}$$

3. The knowledge of the Blue's functions allows for a direct assessment of the structural character of the spectral distribution (energy bands). This is particularly important for systems undergoing a phase change. In general, a given spectrum has a support L defined by the sequence of branch points $\pm A_i, i = 1, \ldots, p$ with positive real A_i and $A_1 > A_2 > \ldots > A_p$ consisting of p arcs, $[-2A_1, -2A_2], \ldots, [2A_2, 2A_1]$ on which the spectrum is non-zero. Since the density of states at the end-point behaves like $\sqrt{A - \lambda}$, its location is determined by the condition

$$\frac{dG}{dz}|_{z=A} = \infty \tag{18}$$

This condition can be translated into a condition on the Blue's function [11]. Let $F(z, B(z))$ be Pastur's equation (15) rewritten in terms of the Blue's function, i.e.

$$F(z, B(z)) \equiv \frac{1}{N} \sum_i \frac{1}{B(z) - \epsilon_i - z} - z = 0 \tag{19}$$

Then the conditions determining the end-points of the spectral distribution follow from the simultaneous solutions to

$$\frac{dF(z, B(z))}{dz} = 0$$

$$F(z, B(z)) = 0 \tag{20}$$

with

$$\frac{dB(z)}{dz}|_A = 0 \tag{21}$$

The latter condition is just the end-point condition (18) rewritten in terms of the Blue's function. The set of all end-points $\pm A_i$ in the spectral distribution comes from solving for B in the above set of equations (20,21).

When Pastur's equation is cubic in G (Cardano class), the corresponding Blue's equation (19) is quadratic, and the end-point conditions (20) follow from the solutions to a linear and biquadratic (quartic) equation, resulting into four roots $\pm A_1, \pm A_2$. Therefore the spectral distribution consists of two arcs. The condition when A_2 vanishes determines a transition point, with the spectral distribution playing the role of an order parameter. Cardano class consists of two phases P_1 and P_2, and the spectral distribution is either supported by one arc (P_1-phase) or two disconnected arcs (P_2-phase).

When Pastur's equation is quartic in G (Ferrari class), the corresponding Blue's equation (19) is cubic, and the boundary conditions reduce to the simultaneous solutions of a quadratic and cubic equation, thereby maximally six roots $\pm A_1, \pm A_2, \pm A_3$. Therefore the support for the spectral distribution consists of either three, two or one arcs, corresponding to the phases P_3, P_2, or P_1. The critical points are determined from the condition that the real positive points A_3 and/or A_2 vanish. We note that the present analysis can be easily extended numerically to higher degree polynomials, in the absence of the explicit Green's functions. This is particularly important for multiflavor QCD.

4. As a direct application of the ideas discussed above, we now consider four different chiral random matrix models as inspired from the symmetries of the QCD Dirac operator and also instanton calculations.

• Lattice with 1 flavor.

Recently, Kalkreuter [10] has performed detailed lattice analysis of the distribution of eigenvalues for the (hermitean) Dirac operator $\mathbf{Q}_1 = \gamma_5(\not{D}+m)$ using Wilson fermions in the quenched approximation. The multiplication by γ_5 implies that $\mathbf{Q}_1^\dagger = \mathbf{Q}_1$. In terms of random matrix ensembles, Wilson fermions are represented by the Gaussian Unitary Ensemble (GUE) for $N_c > 2$ and by the Gaussian Orthogonal Ensemble (GOE) for $N_c = 2$. For one-flavor, we have

$$\mathbf{Q}_1 = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} + \mathbf{R} \ . \tag{22}$$

All the entries in the deterministic matrix are $N \times N$ valued, while the random and hermitean matrix $\mathbf{R}$ is $2N \times 2N$ valued. Because the Wilson r-terms (needed to eliminate the quark doublers) break explicitly chiral symmetry, $\mathbf{R}$ is not block-off diagonal. So while the eigenvalues of $\mathbf{R}$ are not paired, the eigenvalue distribution associated to $\mathbf{R}$ is symmetric about zero virtuality. Randomness populates evenly the positive and negative states. For one flavor and in the massless case, the spectrum is hoped to be $U(1)_L \times U(1)_R$ symmetric on the average. The Green's function of the random matrix in (22) for the Gaussian ensemble is given by (4). The resolvent $G(z)$ for the deterministic and random problem (22) is one of the solution to the equation [4]

$$G^3 - 2zG^2 + (z^2 - m^2 + 1)G - z = 0 \tag{23}$$

Kalkreuter's lattice spectra for one Wilson fermion belong to the Cardano class. A typical spectral distribution following from (23) is shown in Fig. 1. It yields a structural change (transition from P_1 to P_2) for a critical mass $m_* = 1$, with a density of states at zero virtuality $\nu(\lambda) \sim \lambda^{1/3}$ [4]. The localization of the heavy quark, suggests that the flavor symmetry is broken to $U_V(1)$ with the appearance of a heavy quark symmetry, *i.e.* invariance under spin-flip. It should be clear, however, that in our random analysis, (22) does not account for the $U_A(1)$ anomaly. The resulting spectral distribution, however, is qualitativaly similar to the one obtained by Kalkreuter on the lattice using Wilson fermions [4].

In physical units $m_* = 1/\Sigma = 100 - 200$ MeV, which is rather close to the strange quark mass in QCD. The ori-

gin of the scale Σ follows from the Banks-Casher relation. Indeed, for two flavors $< \overline{q}q >= -2\Sigma N/V_4$ [2], where the chiral condensate is $< \overline{q}q >=< \overline{u}u + \overline{d}d >$, with typically $< \overline{u}u >\approx< \overline{d}d >= -((200-250)\ \mathrm{MeV})^3$. The density of quark zero modes is roughly $n = N/V_4 \sim 1\ \mathrm{fm}^{-4}$.

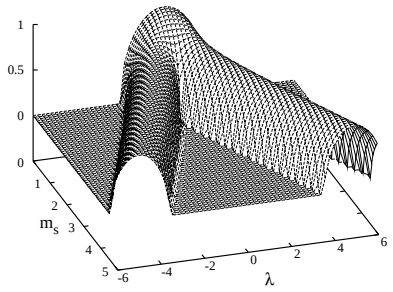

FIG. 1. Spectral distribution for one flavor (Wilson fermions) as a function of the eigenvalue λ and the quark mass m_s.

• Finite Temperature.

Recently, Jackson and Verbaarschot [16] have suggested a simple chiral random matrix model for the massless QCD Dirac operator $i\not{D}$ on the cylinder, namely

$$\mathbf{Q}_2 = \begin{pmatrix} 0 & \pi T \\ \pi T & 0 \end{pmatrix} + \begin{pmatrix} 0 & R \\ R^\dagger & 0 \end{pmatrix} \tag{24}$$

where πT is short for one of the two lowest Matsubara frequencies. Again all entries in (24) are $N \times N$ matrix valued. Although the low temperature phase involves many Matsubara frequencies, (24) may be viewed as a schematic truncation with the right zero and high temperature content [7].

This model belongs to the Cardano class. Indeed, the Green's function for the deterministic (thermal part) could be easily rewritten in a diagonal form by squaring the operator, with the result

$$G_T(z) = \frac{z}{z^2 - (\pi T)^2} \tag{25}$$

Adding the corresponding Blue's function we recover the cubic equation (23) with the substitution $m \to \pi T$. This result is expected from dimensional reduction arguments [17]. At high temperature, and after suitable chiral rotations, each Matsubara mode carries a thermal mass, the lowest being $\mathbf{m}(T) = \sqrt{m^2 + \pi^2 T^2}$. The corresponding spectral distribution has been discussed numerically [16], and analytically [18]. The critical temperature is $T_* = 1/\pi$. In dimensionfull units it is $T_* = 30 - 70$ MeV. Of course, this is rather unreasonable, but so is (24). The addition of more Matsubara modes affects quantitatively T_* [7].

The critical temperature may also be assessed analytically by analyzing the end-points of the spectrum. Indeed, in the Cardano class the end-points $\pm A_1, \pm A_2$ for the thermal problem are located at

$$A_1 = \frac{1}{2^{3/2}\pi T} \frac{(4\pi^2 T^2 - 1 + \sqrt{8\pi^2 T^2 + 1})^{3/2}}{\sqrt{8\pi^2 T^2 + 1} - 1}$$
$$A_2 = \frac{1}{2^{3/2}\pi T} \frac{(4\pi^2 T^2 - 1 - \sqrt{8\pi^2 T^2 + 1})^{3/2}}{\sqrt{8\pi^2 T^2 + 1} + 1} \tag{26}$$

The same equations govern the evolution of the end points in the one-flavor case discussed above and in [4], with the identification $m = \pi T$, as well as the end-points of the energy bands in the case of spin-dependent scattering off impurities in quantum Hall fluids [11]. The real part of A_2 vanishes at the critical point $T_* = 1/\pi$.

• Lattice with $2 + 1$ flavors.

The issue of chiral symmetry breaking in QCD is particularly interesting for the case of two light (up and down) and one heavy (strange) flavors. On the lattice this is usually difficult to achieve with staggered fermions. It is best approached using Wilson fermions. Consider the case where the up and down quark mass are small but the strange quark mass is kept fixed (2+1 flavors). Now, let us consider with Kalkreuter the spectral distribution for the hermitean Dirac operator $\mathbf{Q}_3 = \gamma_5(\not{D} + M)$ with $M = \mathrm{diag}(m_u, m_d, m_s)$ (with $m_u = m_d \approx 0$) and Wilson fermions.

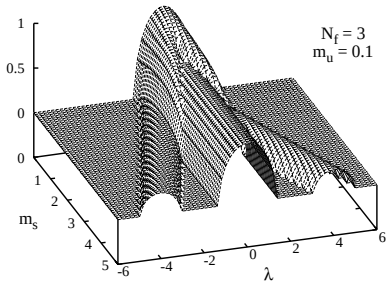

FIG. 2. Spectral distribution for 2+1 flavors (Wilson fermions), as a function of the eigenvalue λ and the strange quark mass m_s.

In this case, the Green's function is given by

$$G(z) = \left\langle \frac{1}{6N} \mathrm{Tr}\, \frac{1}{z - \mathbf{Q}_3} \right\rangle \tag{27}$$

with

$$\mathbf{Q}_3 = \bigotimes_{i=u,d,s} \left(\begin{pmatrix} m_i & 0 \\ 0 & -m_i \end{pmatrix} + \mathbf{R} \right) \tag{28}$$

Each entry in the deterministic matrix is again $N \times N$ valued, while the random matrix is hermitean and $2N \times 2N$ valued. Due to the block-diagonal structure of $\mathbf{Q}_3$, the resolvent (27) is the sum of three resolvents, each of the Cardano type. Specifically,

$$G = \frac{2}{3} G_{m_u} + \frac{1}{3} G_{m_s} \tag{29}$$

where G_m are solutions to (23) with the corresponding masses. Figure 2 shows the corresponding spectral function for two light quarks $m_u = m_d = 0.1$, as a function of the heavy quark mass m_s.

For small values of m_s, the spectrum is $U(3)_L \times U(3)_R$ symmetric, with soft breaking by the up, down and strange quark mass. (The assumption being that the r-terms do not affect considerably the chiral structure in the Wilson formulation). For large values of m_s, the spectrum is $U(2)_L \times U_R(2)$ with an additional invariance under a spin-flip of the strange quark (heavy-quark symmetry). We recall that the critical quark mass for the Cardano class is $m_* = 1$ or $100 - 200$ MeV in physical units. For $m_s > m_*$, the spectral distribution resembles the P_3 phase expected from a quartic equation (Ferrari). However, the two are distinguishable by the evolution of their end-points.

• Instanton Gas-Liquid

In the instanton model of the QCD vacuum, the interaction between the instantons can cause them to cluster or dissociate into a gas, depending on the density of instantons per unit volume. Early simulations using instantons have revealed an admixture of free instantons (gas) and instanton clusters at low instanton density[2]. At a critical density $n_* \sim 1$ fm^{-4} a random phase (intermediate between liquid and gas) is formed, that breaks spontaneously chiral symmetry [19]. Similar ideas have also been used to mock up the chiral transition in the instanton vacuum at finite temperature, using the Euclidean temporal direction as a mean of polarization [20]. For these problems, the generic structure of the Dirac operator truncated to the space of zero modes is

$$\mathbf{Q}_4 = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} + \begin{pmatrix} 0 & R \\ R^\dagger & 0 \end{pmatrix} \tag{30}$$

The matrix D carries off-diagonal short (molecule) or medium (cluster) range order. It is $N \times N$ valued. The matrix elements of D in instanton models are identified with the hopping matrix elements between instantons and antiinstantons, hence chirality odd. For simplicity, we will restrict our discussion to the case where D is diagonal since a molecule only involves one instanton and one anti-instanton at a time. The elements of the matrix D, however, need not be equal.

Equation (30) is the sum of a deterministic D and a random R matrix. The case where $D = \mathrm{diag}(d, d, ..., d)$ yields to the Cardano class discussed above. The case where some fraction α of the elements along the diagonal of D are zero, and all the others are non-zero and equal to d yields a Green's function for the molecules of the form

$$G_{mol}(z) = \alpha \frac{1}{z} + (1-\alpha) \frac{z}{z^2 - d^2} \tag{31}$$

The addition law of the Blue's functions results in a quartic equation. The spectrum follows from the discontinuity of the pertinent solution to

$$G = \alpha \frac{1}{z - G} + (1-\alpha) \frac{z - G}{(z-G)^2 - d^2} \tag{32}$$

The problem is of the Ferrari class. The roots of (32) are known algebraically, so they can be used to derive explicitly the spectral distribution as shown in Fig. 3.

The end-point singularities of (32) are given by

$$A_i = \pm \frac{x_i^2 + x_i(1-d^2) - \alpha d^2}{\sqrt{x_i}(x_i - d^2)} \tag{33}$$

where x_i $(i = 1, 2, 3)$ are the real and positive roots of the cubic equation

$$x^3 - x^2(1 + 2d^2) + x d^2 (d^2 + 3\alpha - 1) - \alpha d^4 = 0 \tag{34}$$

For $\alpha = 2/3$, Figure 3 shows a structural change in the spectral distribution for a hopping strength $d* \approx \sqrt{3}$ or in physical units $d_* = 170 - 340$ MeV.

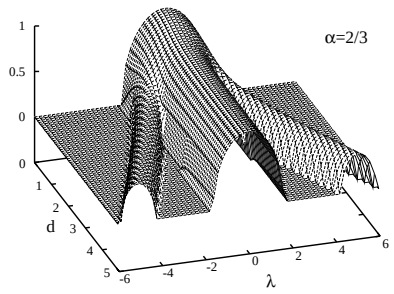

FIG. 3. Spectral distribution for the instanton model (gas-liquid) as a function of the eigenvalue λ and the hopping strength d, with fixed $\alpha = 0.66$.

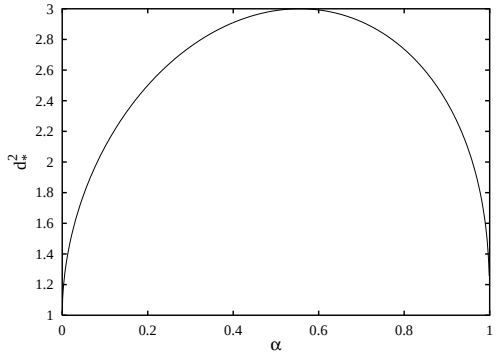

FIG. 4. Behavior of the hopping strength d as a function of α.

[2]We note that this is the same as the density of quark zero modes, because of the Atiyah-Singer index theorem.

The critical value d_* depends on the concentration of molecules $(1-\alpha)$ as shown in Fig. 4. The structural change occurs away from zero virtuality.

In the vicinity of zero, (32) reduces to

$$G^4(0) + (1-d^2)G^2(0) - \alpha d^2 = 0 \tag{35}$$

leading to the condensate (in dimensionless units)

$$|<\bar{q}q>| = \frac{\pi}{\sqrt{2}}\sqrt{1-d^2+\sqrt{(1-d^2)^2+4\alpha d^2}} \tag{36}$$

The sign of the condensate depends on the way we approach zero virtuality in the complex z-plane. This result is similar to the one discussed by Wettig, Schäfer and Weidenmüller [21] using different arguments[3].

The critical points of (32) follow from the behavior of the end-points given by (33-34). In terms of the hopping strength d, the spectral distributions shown in Figs. 5, display novel phase structures as a function of the concentration α. The upper figure corresponds to the case where d is greater than the maximal critical value $d_* = \sqrt{3}$. For $\alpha = 1$ we are in the totally random phase with Wigner's semi-circle. There are no molecules. This is the P_1 phase. For $0 < \alpha < 1$ we have cohabitation phases (P_3 case) between a chirally symmetric molecular phase and a spontaneously broken chiral phase with a non-vanishing chiral condensate. For $\alpha = 0$ all the elements along the diagonal of D are equal to d. In this case the spectral function reduces to the Cardano class, see Fig. 1. The off-diagonal short range order through the molecules (matrix D) is able to destroy the long range coherence produced by the random distribution of R (gas). We are in the P_2 phase. The middle of Fig. 5 shows the spectral distribution for the lowest critical value $d_* = 1$. The lower of Fig. 5 shows the case with $d < d_* = 1$. In the latter, the spectral distribution is always in the P_1 phase. The off-diagonal short range order is not strong enough to destroy the coherence at zero virtuality. In this regime chiral symmetry is always spontaneously broken.

We note that instanton models with an average instanton density of one instanton per fm^4, display spectral distributions that are close to the critical distribution with α small [19]. The instanton-to-molecule model for the finite temperature transition in QCD [20], corresponds to the situation where α interpolates between 1 and 0, discontinuously in the case of a first order transition or smoothly in the case of a second or higher order transition.

[3] The parameter α used in [21] corresponds to our $(1-\alpha)$.

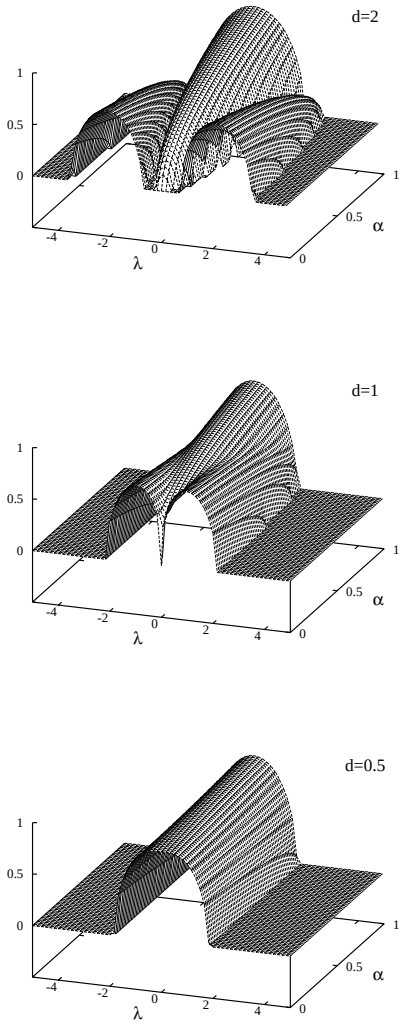

FIG. 5. Spectral distributions for the instanton model as a function of the eigenvalue λ, the concentration α and the hopping strength $d = 2$ (upper), $d = 1$ (middle), and $d = 0.5$ (lower).

5. We have shown how the method of Blue's functions can be applied to a large class of chiral random matrix models. In doing so, we have unfolded two algebraic equations that allow a generic classification of these spectra. Kalkreuter's spectral distribution for one flavor and Wilson fermions, as well as the massless Dirac operator in QCD on the cylinder at high temperature (one-Matsubara), have been shown to fall into the Cardano class (cubic equation). Wilson fermions for 2+1 flavors yield a spectrum that follows from a a linear superposition of Cardano's solutions. Finally, instanton models with off-diagonal short range order fall into the Ferrari class (quartic equation).

In terms of Blue's functions, we have discussed the role played by the end-point singularities and shown how

they could be used to assess the critical parameters. We have suggested that the spectral density of states at the merging end-points (zero-virtuality for instance) could be used as a pertinent order parameter for monitoring phase changes in QCD inspired spectra. This is particularly relevant in the massive case, where the quark condensate ceases to be (strictly speaking) a good order parameter. In doing so, we have unraveled a rich phase structure.

The method we have followed in this paper is very powerful and allows for a number of future investigations. In particular, it allows for a simple assessment of the thermal aspects of the QCD phase diagram for 2+1 flavors using chiral random matrix models. Detailed lattice simulations are by now available to allow for a quantitative understanding in terms of the spectral distributions. This issue will be discussed next.

Acknowledgements

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